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1984 J. Phys. A: Math. Gen. 17 1777

(http://iopscience.iop.org/0305-4470/17/9/013)

J. Phys. A: Math. Gen. 17 (1984) 1777–1786. Printed in Great Britain

SL(3, *R*) realisations and the damped harmonic oscillator

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Received 1 December 1983, in final form 1 March 1984

Abstract. The point and contact Lie groups for the damped harmonic oscillator are found and several realisations of the SL(3, *R*) Lie group are analysed. The non-critical and critical cases are studied separately and a sort of limiting procedure is defined, which leaves the Lie algebra structure unchanged. Some physical applications of these results are also pointed out.

1. Introduction

This paper deals with the complete determination of the group of transformations which leaves invariant the equations of motion and the action of the harmonic oscillator with a damping term. Previous work on the subject which should be mentioned includes a detailed analysis of the harmonic oscillator without friction (Wulfman and Wybourne 1976, Lutzky 1978), the repulsive undamped oscillator (Leach 1980a) and the undamped harmonic oscillator with variable frequency (Prince and Eliezer 1980, Leach 1980b). All these previous cases deal only with point transformations. Contact symmetries for the undamped case were also recently considered (Schwarz 1983). Also, a method using mappings among differential equations through the Arnold transformation (Arnold 1983) was also developed (Martini and Kersten 1983) for the damped harmonic oscillator, but this method relies heavily on the knowledge of the solutions; something which we find unnecessary and cumbersome. Besides, nothing is said in the paper about the group structure; neither is reference made to the group leaving the action invariant: a fact which from our viewpoint has far reaching consequences, specially in the quantum domain, as we shall see later on.

Therefore, we have adopted here the old Lie method (Lie 1894) for the equations of motion and Noether's theorem for the conserved quantities (Noether 1918) for the action. The group structure is analysed throughout as well as the subgroups leaving the action invariant. A distinct difference is made between the over- (under-) damped cases on one hand and the critical case on the other hand. Consequently, § 2 is devoted to the point symmetry group leaving the equations of motion for the first (over- and under-damped) case. Section 3 deals with the same calculation for the second (critical) case. Section 4 contains the symmetry group of the action in both cases and several considerations are made in regard to the Hamiltonian, as well as the conserved quantities. Contact transformations for both cases are the subject of § 5, and a set of functions is given which allows us to recover the previously considered point transformations. Finally, in § 6 we give the necessary and sufficient conditions to obtain the point-Lie group of the critical case as a limiting procedure from the non-critical generators: something which appears to be similar to the so-called group contraction although several crucial differences arise. We close with a few conclusions.

2. Point symmetry group for the non-critical cases

We shall be dealing with the one-dimensional mechanical system given by the Lagrangian

$$\mathscr{L} = e^{bt/m}(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega_0^2 x^2) \tag{1}$$

whose equations of motion are

$$m\ddot{x} + b\dot{x} + m\omega_0^2 x = 0. \tag{2}$$

Since the $\dim[b] = \mathrm{MT}^{-1}$ we can form the dimensionless quantity

$$\gamma_0 = b/m\omega_0 \tag{3}$$

and rescaling the time variable in a dimensionless way by putting

$$\tau = \omega_0 t \tag{4}$$

we end up with a Lagrangian

$$\mathscr{L} = \{\tfrac{1}{2}m\omega_0^2\}\, e^{\gamma_0\tau}[(\mathrm{d}x/\mathrm{d}\tau)^2 - x^2] \tag{5}$$

and an equation of motion

$$\mathrm{d}^2x/\mathrm{d}\tau^2 + \gamma_0\,\mathrm{d}x/\mathrm{d}\tau + x = 0. \tag{6}$$

Also, we shall use the following quantity

$$\gamma = (\gamma_0^2 - 4)^{1/2}. \tag{7}$$

The non-critical cases arise from $\gamma \neq 0$ (real or pure imaginary). The critical case is $\gamma = 0$ ($\gamma_0 = 2$). Also, the harmonic oscillator limit is obtained for $\gamma_0 = 0$ ($\gamma = 2\mathrm{i}$).

The use of Lie theory in finding the group of transformations which leaves invariant a differential equation is well known (Lie 1894, Lutzky 1978) and we shall not give here the detailed steps. The result is a set of eight generators given by the following expressions

$$\begin{aligned}
&G_1 = (2/\gamma)[\sinh(\gamma\tau)\partial/\partial\tau + (\tfrac{1}{2}\gamma\cosh(\gamma\tau) - \tfrac{1}{2}\gamma_0\sinh(\gamma\tau))x\partial/\partial x]\\
&G_2 = (2\mathrm{i}/\gamma)[\cosh(\gamma\tau)\partial/\partial\tau + (\tfrac{1}{2}\gamma\sinh(\gamma\tau) - \tfrac{1}{2}\gamma_0\cosh(\gamma\tau))x\partial/\partial x]\\
&G_3 = (2\mathrm{i}/\gamma)(e^{-\gamma_0\tau/2}\cosh(\tfrac{1}{2}\gamma\tau)\partial/\partial x) \qquad G_4 = (2/\gamma)(e^{-\gamma_0\tau/2}\sinh(\tfrac{1}{2}\gamma\tau)\partial/\partial x)\\
&G_5 = (2\mathrm{i}/\gamma)(\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x\partial/\partial x) \qquad G_6 = x\partial/\partial x\\
&G_7 = -\mathrm{i}e^{\gamma_0\tau/2}[x\sinh(\tfrac{1}{2}\gamma\tau)\partial/\partial\tau + (\tfrac{1}{2}\gamma\cosh(\tfrac{1}{2}\gamma\tau) - \tfrac{1}{2}\gamma_0\sinh(\tfrac{1}{2}\gamma\tau))x^2\partial/\partial x)\\
&G_8 = e^{\gamma_0\tau/2}[x\cosh(\tfrac{1}{2}\gamma\tau)\partial/\partial\tau + (\tfrac{1}{2}\gamma\sinh(\tfrac{1}{2}\gamma\tau) - \tfrac{1}{2}\gamma_0\cosh(\tfrac{1}{2}\gamma\tau))x^2\partial/\partial x].
\end{aligned} \tag{8}$$

Notice that time translation is in fact a generator belonging to the symmetry group of the equations of motion since

$$\partial/\partial\tau = (\gamma/2\mathrm{i})[G_5 + \mathrm{i}(\gamma_0/\gamma)G_6] = (\gamma/2\mathrm{i})G_5 + \tfrac{1}{2}\gamma_0 G_6 \tag{9}$$

is a linear combination of G_5 and G_6. This will be proven to be not true for the group leaving the action invariant. The expressions (8) are written for γ real (over-damped oscillator). However, for γ pure imaginary we should take into account that $\sinh(\mathrm{i}\alpha) = \mathrm{i}\sin(\alpha)$ and $\cosh(\mathrm{i}\alpha) = \cos\alpha$ and we shall obtain the other non-critical case (the under-damped oscillator). The commutation rules of (8) are

$$
\begin{aligned}
&[G_1, G_2] = -2G_5 \qquad [G_5, G_1] = 2G_2 \qquad [G_2, G_5] = 2G_1 \qquad [G_3, G_4] = 0 \\
&[G_3, G_1] = [G_2, G_4] = [G_5, G_4] = G_3 \qquad [G_1, G_4] = [G_3, G_5] = [G_2, G_3] = G_4
\end{aligned} \tag{10}
$$

and also

$$
\begin{aligned}
&[G_6, G_7] = G_7 \qquad [G_6, G_8] = G_8 \qquad [G_7, G_8] = 0 \\
&[G_6, G_1] = [G_6, G_2] = [G_6, G_5] = 0 \qquad [G_6, G_3] = -G_3 \qquad [G_6, G_4] = -G_4 \\
&[G_7, G_1] = -G_7 \qquad [G_7, G_2] = -G_8 \qquad [G_7, G_3] = -\tfrac{1}{2}(G_1 + 3G_6) \\
&[G_7, G_4] = \tfrac{1}{2}(G_2 - G_5) \qquad [G_7, G_5] = -G_8 \\
&[G_8, G_1] = G_8 \qquad [G_8, G_2] = -G_7 \qquad [G_8, G_5] = G_7 \\
&[G_8, G_3] = -\tfrac{1}{2}(G_2 + G_5) \qquad [G_8, G_4] = -\tfrac{1}{2}(G_1 - 3G_6).
\end{aligned} \tag{11}
$$

A comparison with formulae (25) and (44) of Lutzky (1978) shows that this is also the Lie algebra of SL(3, ***R***). Taking the harmonic oscillator limit $\gamma_0 = 0$, our generators reduce to the ones found by this author ($m = 2$, $\omega_0 = 1$) in their formulae (24) and (41)–(43).

3. Point symmetry group for the critical case

In the case $\gamma = 0$, and applying Lie theory, we find the following set of generators leaving the equation of motion invariant:

$$
\begin{aligned}
&C_1 = \partial/\partial\tau - \tfrac{1}{2}\gamma_0 x\partial/\partial x \qquad C_2 = \mathrm{e}^{-\gamma_0\tau/2}\partial/\partial x \qquad C_3 = \mathrm{e}^{-\gamma_0\tau/2}\tau\partial/\partial x \\
&C_4 = \tau^2\partial/\partial\tau + (1 - \tfrac{1}{2}\gamma_0\tau)\tau x\partial/\partial x \qquad C_5 = 2\tau\partial/\partial\tau + (1 - \gamma_0\tau)x\partial/\partial x \\
&C_6 = x\partial/\partial x \qquad C_7 = \mathrm{e}^{\gamma_0\tau/2}(x\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x^2\partial/\partial x) \\
&C_8 = \mathrm{e}^{\gamma_0\tau/2}[\tau x\partial/\partial\tau + (1 - \tfrac{1}{2}\gamma_0\tau)x^2\partial/\partial x].
\end{aligned} \tag{12}
$$

The alert reader will certainly notice that if $\gamma = 0$ then $\gamma_0 = 2$. However we can still leave γ_0 as a free parameter and (12) remains a set of generators leaving invariant the equation for the critical case. They even close a Lie algebra for arbitrary γ_0!. The commutation rules are:

$$
\begin{aligned}
&[C_1, C_2] = [C_2, C_3] = [C_3, C_4] = 0 \qquad [C_1, C_5] = 2C_1 \qquad [C_3, C_5] = -C_3 \\
&[C_1, C_3] = C_2 \qquad [C_2, C_4] = C_3 \qquad [C_4, C_5] = -2C_4 \qquad [C_1, C_4] = C_5 \\
&[C_2, C_5] = C_2
\end{aligned} \tag{13}
$$

and also

$$
\begin{aligned}
&[C_1, C_6]=[C_1, C_7]=[C_4, C_6]=[C_5, C_6]=[C_4, C_8]=0 \qquad [C_2, C_6]=C_2\\
&[C_2, C_7]=C_1 \qquad [C_1, C_8]=C_7 \qquad [C_2, C_8]=\tfrac{1}{2}(C_5+3C_6)\\
&[C_3, C_7]=\tfrac{1}{2}(C_5-3C_6) \qquad [C_3, C_6]=C_3 \qquad [C_4, C_5]=-2C_4\\
&[C_3, C_8]=C_4 \qquad [C_4, C_7]=-C_8 \qquad [C_5, C_7]=-C_7 \qquad [C_6, C_7]=C_7\\
&[C_5, C_8]=C_8 \qquad [C_6, C_8]=C_8.
\end{aligned}
\tag{14}
$$

The time translation is also a symmetry as in the previous (non-critical) case. In fact

$$\partial/\partial\tau = C_1+\tfrac{1}{2}\gamma_0 C_6.$$

However, this will not be true for the group leaving the action invariant. Although the commutation rules (13) and (14) look very different from the ones of (10) and (11), the formal equivalence:

$$
\begin{aligned}
&C_1\Rightarrow\tfrac{1}{2}(G_5+G_2) \qquad C_2\Rightarrow G_3 \qquad C_3\Rightarrow G_4 \qquad C_4\Rightarrow\tfrac{1}{2}(G_5-G_2)\\
&C_5\Rightarrow G_1 \qquad C_6\Rightarrow G_6 \qquad C_7\Rightarrow G_8 \qquad C_8\Rightarrow G_7
\end{aligned}
\tag{15}
$$

reproduces (10) and (11) using (13) and (14). Therefore the structure of the Lie algebra in the critical case is also the one of SL(3, $\boldsymbol{R}$). This result is highly non-trivial since we would expect in principle this structure to change from the fact that the C-generators can be obtained as a series expansion in γ from the G-generators: a sort of group contraction. This is not, however, the case and this result will be carefully analysed in § 6.

4. Point symmetry group for the action

Let S be a definite infinitesimal generator leaving the action invariant. Then, the Killing equation must follow:

$$(S^1)^L\mathscr{L}+\mathscr{L}(\partial\xi/\partial\tau)=\mathrm{d}\Lambda/\mathrm{d}\tau=(\partial/\partial\tau+\dot{x}\partial/\partial x)\Lambda(x,\tau) \tag{16}$$

where S is of the form:

$$S=\xi(x,\tau)\partial/\partial\tau+\eta(x,\tau)\partial/\partial x \tag{17}$$

and S^1 is defined as

$$S^1=\xi(x,\tau)\frac{\partial}{\partial\tau}+\eta(x,\tau)\frac{\partial}{\partial x}+\left(\frac{\mathrm{d}\eta(x,\tau)}{\mathrm{d}\tau}-\dot{x}\frac{\mathrm{d}\xi(x,\tau)}{\mathrm{d}\tau}\right)\frac{\partial}{\partial\dot{x}}. \tag{18}$$

Applying this well known technique (Lutzky 1978) to our case, we find the following.

(a) If $\gamma\neq 0$, there are five generators leaving the action invariant.

They are exactly: G_1, G_2, G_3, G_4 and G_5 given in (8) with commutation rules given by (10). The correspondent conserved quantities are

$$
\begin{aligned}
I_1=\mathrm{e}^{\gamma_0\tau}[&(\dot{x}^2+x^2)\sinh(\gamma\tau)-x\dot{x}(\gamma\cosh(\gamma\tau)-\gamma_0\sinh(\gamma\tau))\\
&+\tfrac{1}{2}x^2(\gamma^2\sinh(\gamma\tau)-\gamma\gamma_0\cosh(\gamma\tau))]
\end{aligned}
$$

$$I_2 = e^{\gamma_0\tau}[(\dot{x}^2 + x^2)\cosh(\gamma\tau) - x\dot{x}(\gamma\sinh(\gamma\tau) - \gamma_0\cosh(\gamma\tau)) + \tfrac{1}{2}x^2(\gamma^2\cosh(\gamma\tau) - \gamma\gamma_0\sinh(\gamma\tau))]$$

$$\begin{aligned} I_3 &= e^{\lambda_2\tau}(\dot{x} + \lambda_1 x) \qquad & \lambda_1 &= \tfrac{1}{2}(\gamma_0 - \gamma) \\ I_4 &= e^{\lambda_1\tau}(\dot{x} + \lambda_2 x) \qquad \text{where} & \lambda_2 &= \tfrac{1}{2}(\gamma_0 + \gamma) \\ I_5 &= e^{\gamma_0\tau}(\dot{x}^2 + x^2 + \gamma_0 x\dot{x}) \end{aligned} \tag{19}$$

and, indeed, they verify

$$I_1 = \tfrac{1}{2}(I_3^2 - I_4^2) \qquad I_2 = \tfrac{1}{2}(I_3^2 + I_4^2) \qquad I_5 = I_3 I_4. \tag{20}$$

The relations (20) are due to the obvious fact of having too many constants of motion for a mechanical system with only one degree of freedom. Then, we can generate the solutions with only two of those constants; say I_3 and I_4 (Gettys *et al* 1981). Notice that time translation is not a symmetry of the action and then the usual energy is not conserved. Instead we have a conservation law for the pseudoenergy given by the G_5-generator and the I_5 constant of motion.

(b) If $\gamma = 0$, we also have five generators leaving the action invariant.

They are C_1, C_2, C_3, C_4 and C_5 in (12) and they close under commutation rules given by (13). The correspondent conserved quantities are

$$\begin{aligned} J_1 &= e^{\gamma_0\tau}(\dot{x}^2 + x^2 + \gamma_0 x\dot{x}) \qquad J_2 = e^{\gamma_0\tau/2}(\dot{x} + \tfrac{1}{2}\gamma_0 x) \\ J_3 &= e^{\gamma_0\tau/2}[\tau(\dot{x} + \tfrac{1}{2}\gamma_0 x) - x] \\ J_4 &= e^{\gamma_0\tau}[\tau^2(\dot{x}^2 + x^2 + \gamma_0 x\dot{x}) + x^2(1 - \gamma_0\tau) - 2x\dot{x}\tau] \\ J_5 &= e^{\gamma_0\tau}[\tau(\dot{x}^2 + x^2 + \gamma_0 x\dot{x}) - \tfrac{1}{2}\gamma_0 x^2 - x\dot{x}]. \end{aligned} \tag{21}$$

In addition, if $\gamma_0 = 2$, as is the case, we obtain the following relations

$$J_1 = J_2^2 \qquad J_4 = J_3^2 \qquad J_5 = J_2 J_3. \tag{22}$$

The condition $\gamma_0 = 2$ is only necessary for the constants of motion but not for the Lie algebra. This suggests a dilatational invariance, in the critical case, under arbitrary reparametrisation of γ_0.

Since we do not have invariance under time translation for the action either in (a) or (b) we are tempted to suggest that, in the quantum domain, the physical operators should only be the ones found in this section. The reason is obviously that classical trajectories have no meaning in quantum mechanics but the action is crucial for calculating the Feynman Green function. The symmetries of the action are much more important in quantum mechanics than the symmetries of the equations of motion. This line of reasoning is being pursued now in a wide class of models to be reported elsewhere.

5. Contact transformations

Recently, we have witnessed a revival in the interest in contact transformation for differential equations of various kinds, which were first developed by Lie (1894) and, later on, the technique was made more transparent by Campbell (1903). These transformations were applied to kinematics in special relativity by the author (Boya and Cerveró 1975a, b, Cerveró 1977a, b). In this section we shall treat the case under consideration by using the well known Lie techniques applied to find the group of contact transformations which leaves invariant the equation (6). The contact group may be of interest mainly because of the field-theoretical conjecture which says that all complete integrable systems have an associated infinite parameter Lie contact group. The contact group for the harmonic oscillator has already been found (Schwarz 1983) and turns out to be an infinite parameter Lie group. We shall follow the steps of the Schwarz paper with differences in the final form of the characteristic function due to the fact that we have benefited from a slightly more advantageous choice of the characteristic curves.

Let S_c be an infinitesimal contact transformation:

$$S_c=\left(\frac{\partial W}{\partial p}\right)\frac{\partial}{\partial\tau}+\left(p\frac{\partial W}{\partial p}-W\right)\frac{\partial}{\partial x}-\left(\frac{\partial W}{\partial\tau}+p\frac{\partial W}{\partial x}\right)\frac{\partial}{\partial p}$$
$$-\left(\frac{\partial^2 W}{\partial\tau^2}+2p\frac{\partial^2 W}{\partial\tau\,\partial x}+p^2\frac{\partial^2 W}{\partial x^2}+2q\frac{\partial^2 W}{\partial\tau\,\partial p}+2qp\frac{\partial^2 W}{\partial x\,\partial p}+q^2\frac{\partial^2 W}{\partial p^2}+q\frac{\partial W}{\partial x}\right)\frac{\partial}{\partial q} \tag{23}$$

where $p=\dot{x}$; $q=\ddot{x}$ and W is the characteristic generating function. In our first (over-(under-) damped) case

$$S_c(q+\gamma_0 p+x)=0 \tag{24}$$

which yields a partial differential equation for W of the form

$$\frac{\partial^2 W}{\partial\tau^2}+p^2\frac{\partial^2 W}{\partial x^2}+(\gamma_0 p+x)^2\frac{\partial^2 W}{\partial p^2}+2p\frac{\partial^2 W}{\partial x\,\partial\tau}-2(\gamma_0 p+x)\frac{\partial^2 W}{\partial p\,\partial\tau}$$
$$-2(\gamma_0 p+x)p\frac{\partial^2 W}{\partial x\,\partial p}+W-p\frac{\partial W}{\partial p}-x\frac{\partial W}{\partial x}+\gamma_0\frac{\partial W}{\partial\tau}=0. \tag{25}$$

The characteristic curves of this hyperbolic-type equation are

$$\begin{aligned}u&=\mathrm{e}^{\lambda_2\tau}(p+\lambda_1 x)\\ v&=\mathrm{e}^{\lambda_2\tau}(p+\lambda_2 x)\end{aligned}\qquad\text{where}\qquad\begin{aligned}\lambda_1&=\tfrac{1}{2}(\gamma_0-\gamma)\\ \lambda_2&=\tfrac{1}{2}(\gamma_0+\gamma)\end{aligned}. \tag{26}$$

Then, using new coordinates: (u, v, τ), we find upon substitution in (25):

$$\partial^2 W/\partial\tau^2+(\lambda_1+\lambda_2)\partial W/\partial\tau+W=0 \tag{27}$$

whose general solution is:

$$W=\Delta_1(u, v)\mathrm{e}^{-\lambda_1\tau}+\Delta_2(u, v)\mathrm{e}^{-\lambda_2\tau} \tag{28}$$

with Δ_1 and Δ_2 arbitrary functions of u and v. Therefore W depends upon arbitrary functions and the contact Lie group has an infinite number of parameters. From (28) one can obtain the general form of S_c using (23). This gives

$$S_c=\left\{\frac{\partial\Delta_1}{\partial u}\exp[(\lambda_2-\lambda_1)\tau]+\frac{\partial\Delta_2}{\partial v}\exp[(\lambda_1-\lambda_2)\tau]+\frac{\partial\Delta_2}{\partial u}+\frac{\partial\Delta_1}{\partial v}\right\}\frac{\partial}{\partial\tau}$$
$$+\left\{p\left[\frac{\partial\Delta_1}{\partial u}\exp[(\lambda_2-\lambda_1)\tau]+\frac{\partial\Delta_2}{\partial v}\exp[(\lambda_1-\lambda_2)\tau]+\frac{\partial\Delta_2}{\partial u}+\frac{\partial\Delta_1}{\partial v}\right]\right.$$

$$
\left.-\Delta_1\exp(-\lambda_1\tau)-\Delta_2\exp(-\lambda_2\tau)\right\}\frac{\partial}{\partial x}
$$

$$
+\left\{\lambda_1\Delta_1\exp(-\lambda_1\tau)+\lambda_2\Delta_2\exp(-\lambda_2\tau)\right.
$$

$$
-(\gamma_0 p+x)\left[\frac{\partial\Delta_1}{\partial u}\exp[(\lambda_2-\lambda_1)\tau]+\frac{\partial\Delta_2}{\partial v}\exp[(\lambda_1-\lambda_2)\tau]\right.
$$

$$
\left.\left.+\frac{\partial\Delta_1}{\partial v}+\frac{\partial\Delta_2}{\partial u}\right]\right\}\frac{\partial}{\partial p}. \tag{29}
$$

For specific forms of Δ_1 and Δ_2 we can reobtain the point-Lie group already analysed in § 2. These choices are

$$
\begin{array}{c|c|c|c|c|c|c|c|c}
 & G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 & G_8 \\
\hline
\Delta_1 & (1/\gamma)u & (\mathrm{i}/\gamma)u & -(\mathrm{i}/\gamma) & (1/\gamma) & (\mathrm{i}/\gamma)v & -(1/\gamma)v & -(\mathrm{i}/\gamma)(uv+v^2) & (1/\gamma)(uv-v^2) \\
\hline
\Delta_2 & -(1/\gamma)v & (\mathrm{i}/\gamma)v & -(\mathrm{i}/\gamma) & -(1/\gamma) & (\mathrm{i}/\gamma)u & (1/\gamma)u & (\mathrm{i}/\gamma)(uv+u^2) & (1/\gamma)(uv-u^2)
\end{array} \tag{30}
$$

Turning to the second (critical) case, we find the following set of characteristic curves

$$
u=\mathrm{e}^{\gamma_0\tau/2}(p+\tfrac{1}{2}\gamma_0 x) \qquad v=\mathrm{e}^{\gamma_0\tau/2}[\tau(p+\tfrac{1}{2}\gamma_0 x)-x]. \tag{31}
$$

The change to new coordinates (u, v, τ) yields a characteristic function of the form

$$
W=(\Delta_1(u,v)+\tau\Delta_2(u,v))\mathrm{e}^{-\gamma_0\tau/2} \tag{32}
$$

and a general infinitesimal contact generator S_c:

$$
S_c=\left(\frac{\partial\Delta_1}{\partial u}+\tau\frac{\partial\Delta_1}{\partial v}+\tau\frac{\partial\Delta_2}{\partial u}+\tau^2\frac{\partial\Delta_2}{\partial v}\right)\frac{\partial}{\partial\tau}+\left\{p\left[\frac{\partial\Delta_1}{\partial u}+\tau\frac{\partial\Delta_1}{\partial v}+\tau\frac{\partial\Delta_2}{\partial u}+\tau^2\frac{\partial\Delta_2}{\partial v}\right]\right.
$$

$$
\left.-(\Delta_1+\tau\Delta_2)\,\mathrm{e}^{-\gamma_0\tau/2}\right\}\frac{\partial}{\partial x}+\left\{[\tfrac{1}{2}\gamma_0(\Delta_1+\tau\Delta_2)-\Delta_2]\,\mathrm{e}^{-\gamma_0\tau/2}\right.
$$

$$
\left.+(\gamma_0 p+x)\left[\frac{\partial\Delta_1}{\partial u}+\tau\frac{\partial\Delta_1}{\partial v}+\tau\frac{\partial\Delta_2}{\partial u}+\tau^2\frac{\partial\Delta_2}{\partial v}\right]\right\}\frac{\partial}{\partial p}. \tag{33}
$$

Also, for specific forms of Δ_1 and Δ_2 we reobtain the point-Lie group already analysed in § 3.

$$
\begin{array}{c|c|c|c|c|c|c|c|c}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \\
\hline
\Delta_1 & u & -1 & 0 & 0 & v & v & -uv & -v^2 \\
\hline
\Delta_2 & 0 & 0 & -1 & v & u & -u & u^2 & uv
\end{array}. \tag{34}
$$

6. Two realisations of SL(3, *R*)

The infinitesimal generators (8), with commutation rules (10)–(11), which correspond to the $\gamma\neq 0$ case, and the generators (12), with commutation rules (13)–(14), for the $\gamma=0$ case, represent two different, but not unrelated, realisations of SL(3, *R*). It is the purpose of this section to elucidate such a relationship between these two different realisations. Naively, we would expect a sort of group contraction (Wigner and Inönü 1953, 1954) to work in the $\gamma \to 0$ limit. This cannot be the case since both Lie algebras have the same structure; a fact which contradicts the main properties of the group contraction. Therefore a more careful analysis of the $\gamma \to 0$ limit needs to be carried out. To this end, we expand the generators G of (8) in powers of γ and keeping the leading terms in γ^{-1} and γ, we obtain

$$
\begin{aligned}
&G_1 = [2\tau\partial/\partial\tau + (1-\gamma_0\tau)x\partial/\partial x] + O(\gamma^2)\\
&G_2 = (2i/\gamma)(\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x\partial/\partial x) + i\gamma[\tau^2\partial/\partial\tau + (1-\tfrac{1}{2}\gamma_0\tau)\tau x\partial/\partial x] + O(\gamma^2)\\
&G_3 = (2i/\gamma)\,(e^{-\gamma_0\tau/2}\partial/\partial x) + \tfrac{1}{2}i\gamma\,(\tfrac{1}{2}e^{-\gamma_0\tau/2}\tau^2\partial/\partial x) + O(\gamma^2)\\
&G_4 = (e^{-\gamma_0\tau/2}\tau\partial/\partial x) + O(\gamma^2) \qquad G_5 = (2i/\gamma)(\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x\partial/\partial x) + O(\gamma^2)\\
&G_6 = (x\partial/\partial x) \qquad G_7 = -\tfrac{1}{2}i\gamma\{e^{\gamma_0\tau/2}[\tau x\partial/\partial\tau + (1-\tfrac{1}{2}\gamma_0\tau)x^2\partial/\partial x]\} + O(\gamma^2)\\
&G_8 = [e^{\gamma_0\tau/2}(x\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x^2\partial/\partial x)] + O(\gamma^2).
\end{aligned}
\tag{35}
$$

Inspection of (35) shows that the generators G_2 and G_3 contain more than one term in the expansion and also the generators G_2 and G_5 contain repeated terms. These features act together in such a way that if we naively drop terms in γ with respect to those in $(1/\gamma)$ (for $\gamma \to 0$), the second term in G_2 is lost. This is not, however, the case if we first take linear combinations in the form

$$
\begin{aligned}
&\tfrac{1}{2}(G_5 + G_2) = (2i/\gamma)\{\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x\partial/\partial x\} + \tfrac{1}{2}i\gamma[\tau^2\partial/\partial\tau + (1-\tfrac{1}{2}\gamma_0\tau)\tau x\partial/\partial x]\\
&\tfrac{1}{2}(G_5 - G_2) = -\tfrac{1}{2}i\gamma[\tau^2\partial/\partial\tau + (1-\tfrac{1}{2}\gamma_0\tau)\tau x\partial/\partial x]
\end{aligned}
\tag{36}
$$

and one can take now the limit $\gamma \to 0$ without losing the above mentioned generator. We can also observe that this procedure cannot be made to work for the *second term* in G_3 which is unique in the expansion. Thus, *this is a spurious generator which can be dropped* with respect to the first term in G_3. Now, we see that the so-called 'formal equivalence' (15) has an origin deeply enrooted in the limiting procedure $\gamma \to 0$.

There is another (and perhaps more transparent) way to understand both the 'formal equivalence' (15) and the limiting procedure relating the two different $SL(3, \boldsymbol{R})$-realisations ((8) and (12)). It is based on the observation that, *even for* $\gamma \neq 0$, the operators G_1, $(\gamma/4i)(G_5 + G_2)$, $(i/\gamma)(G_5 - G_2)$, $(\gamma/2i)G_3$, G_4, G_6, $(2i/\gamma)G_7$ and G_8 exist and are perfectly good generators for the invariance algebra of the non-critical oscillator; they are merely linear combinations of the original G's. Now, using the commutation relations (10)–(11) one can easily prove that these operators satisfy exactly the same commutation relations as C_5, C_1, C_4, C_2, C_3, C_6, C_8 and C_7 respectively (see (13)–(14)). This can be done without taking any particular limit for γ. Furthermore, they give precisely the C-operators in the $\gamma \to 0$ limit, as can easily be seen from our expansion (35).

We would like to end with a remark on the structure of two interesting subgroups of the G (or C) group. Firstly, we make the observation that as opposed to the classical case in which the equations of motions (and trajectoria) are fundamental, the group leaving the action invariant is what really matters in the quantum theory. Thus, the important group in the quantum domain would be $\{G_1, G_2, G_3, G_4 \text{ and } G_5\}$ or $\{C_1, C_2, C_3, C_4 \text{ and } C_5\}$ by means of the equivalence (15). Notice that there is not a true 'Hamiltonian' here; rather we have a 'pseudo-Hamiltonian' generator G_5 (or C_1)

$$`H' \equiv (\partial/\partial\tau - \tfrac{1}{2}\gamma_0 x \partial/\partial x). \tag{37}$$

If we then use group theory as a guidance for the right choice of physical observables (rather than use for instance the ambiguous 'correspondence principle'), we should choose this generator as a quantum operator instead of the usual time translation. We have found similar situations in other field-theoretical systems which will be reported elsewhere, but we would like to point out that even in simple mechanical systems, such as the one analysed here, a similar situation arises which suggests that perhaps the problem of choosing physical operators in quantum mechanics should be completely rethought using invariant-action generators.

Another interesting remark is the one related to the spurious generator appearing in the expansions of G_3

$$E = \{\tfrac{1}{2}\, e^{-\gamma_0\tau/2}\tau^2\partial/\partial x\}. \tag{38}$$

This generator closes a Lie algebra with some of the elements of the C-group. More specifically

$$\begin{array}{llll} [C_1, C_2] = 0 & [C_1, C_3] = C_2 & [C_1, C_5] = 2C_1 & [C_1, C_6] = 0 \\ [C_2, C_3] = 0 & [C_2, C_5] = C_2 & [C_2, C_6] = C_2 & [C_3, C_5] = -C_3 \\ [C_3, C_6] = C_3 & [C_5, C_6] = 0 & & \end{array} \tag{39}$$

and also

$$[E, C_1] = -C_3 \qquad [E, C_2] = [E, C_3] = 0 \qquad [E, C_5] = -3E \qquad [E, C_6] = E.$$

Thus, we have the interesting group $\{C_1, C_2, C_3, C_5, C_6, E\}$ as well as the invariant-action subgroup $\{C_1, C_2, C_3, C_4, C_5\}$. The structure of these Lie algebras is now being analysed and their physical properties will be the subject of a forthcoming publication.

Acknowledgments

One of us (JC) is indebted to Dr M Santander for several illuminating conversations.

We also would like to thank the referee for several remarks concerning the limiting procedure which have been useful for clarifying the subject as well as for pointing out several references which were missed in the first version of this paper.

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